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Quantum integrable systems and Clebsch–Gordan series: II

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Abstract. The class of quantum integrable systems associated with root systems was introduced (Olshanetsky M A and Perelomov A M 1977 *Lett. Math. Phys.* **2** 7–13) as a generalization of the Calogero–Sutherland systems (Calogero F 1971 *J. Math. Phys.* **12** 419–36, Sutherland B 1972 *Phys. Rev.* A **4** 2019–21). For the potential $v(q) = \kappa(\kappa - 1)\sin^{-2} q$, the wavefunctions of such systems are related to polynomials in l variables (l is a rank of the root system) and are a generalization of Gegenbauer polynomials and Jack polynomials (Jack H 1970 *Proc. R. Soc. Edinburgh* A **69** 1–18). In Perelomov A M 1998 *J. Phys. A: Math. Gen.* **31** L31–7, it was proved that the series for the product of two such polynomials is a κ-deformation of the Clebsch–Gordan series. This yields recurrence relations for these polynomials and, in particular, for generalized zonal polynomials on symmetric spaces.

This paper follows my paper mentioned above and also Perelomov A M, Ragoucy E and Zaugg Ph 1998 *J. Phys. A: Math. Gen.* **31** L559–65. In the latter, the recurrence relations were used to compute the explicit expressions for A_2-type polynomials, i.e., for the wavefunctions of the three-body Calogero–Sutherland system.

As shown by Ragoucy, Zaugg and Perelomov (see the appendix), similar results are also valid in the A_2 case for the more general two-parameter deformation ((q, t)-deformation) introduced by Macdonald (Macdonald I G 1988 Orthogonal polynomials associated with root systems *Preprint*).

1. Introduction

The class of quantum integrable systems associated with root systems was introduced in [1] (see also [8, 9]) as a generalization of the Calogero–Sutherland systems [2, 3]. Such systems depend on one real parameter κ (for root systems of the type A_n, D_n and E_6, E_7, E_8), on two parameters (for B_n, C_n, F_4 and G_2) and on three parameters for the BC_n. These parameters are related to the coupling constants of the quantum system.

For the potential $v(q) = \kappa(\kappa - 1)\sin^{-2} q$ and special values of parameter κ, the wavefunctions correspond to the characters of the compact simple Lie groups ($\kappa = 1$) [10, 11] or to zonal spherical functions on symmetric spaces ($\kappa = \frac{1}{2}, 2, 4$) [12, 13]. At arbitrary values of κ, they provide an interpolation between these objects.

This class has many remarkable properties. Here we mention only one: the wavefunctions of such systems are a natural generalization of special functions (hypergeometric functions) to the case of several variables. The history of this problem and some results may be found in [9]. In [5], it was shown that the product of two wavefunctions is a finite linear combination of analogous functions, namely, of functions that appear in the corresponding Clebsch–Gordan series. In other words, this deformation (κ-deformation) does not change the Clebsch–Gordan series. For rank 1, we obtain the well known cases of the Legendre, Gegenbauer

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and Jacobi polynomials, and the limiting cases of the Laguerre and Hermite polynomials (see, for example, [14]). Some other cases were also considered in [4,7,15–32]. In [6], a κ-deformed Clebsch–Gordan series was used in order to obtain the explicit expressions for the generalized Gegenbauer polynomials† of type A_2 which is what gives the explicit solution of the three-body Calogero–Sutherland model. For special values of $\kappa = \frac{1}{2}, 2, 4$, these formulae give the explicit expressions for zonal polynomials of type A_2.

In the appendix, analogous results obtained by Ragoucy, Zaugg and Perelomov for the two-parameter family of polynomials of type A_2 introduced by Ruijsenaars [33] and Macdonald [7] are presented.

2. General description

The systems under consideration are described by the Hamiltonian (for more details, see [9])

$$H = \tfrac{1}{2}p^2 + U(q) \qquad p^2 = (p, p) = \sum_{j=1}^{l} p_j^2 \tag{2.1}$$

where $p = (p_1, \ldots, p_l)$, $p_j = -\mathrm{i}\partial/\partial q_j$, is a momentum operator, and $q = (q_1, \ldots, q_l)$ is a coordinate vector in the l-dimensional vector space $V \sim \mathbb{R}^l$ with standard scalar product $(\cdot, \cdot)$. The potential $U(q)$ is constructed by means of a certain system of vectors $R^+ = \{\alpha\}$ in V (the so-called *root systems*):

$$U = \sum_{\alpha\in R^+} g_\alpha^2 v(q_\alpha) \qquad q_\alpha = (\alpha, q) \qquad g_\alpha^2 = \kappa_\alpha(\kappa_\alpha - 1) \qquad g_\alpha = g_\beta \quad \text{if} \quad (\alpha, \alpha) = (\beta, \beta). \tag{2.2}$$

Such systems are completely integrable for potentials of five types (see [9] for A_l; [24–27, 29] for a general case). They are a generalization of the Calogero–Sutherland systems [2, 3] for which $\{\alpha\} = \{e_i - e_j\}$, $\{e_j\}$ being a standard basis in V.

In this paper, we consider in detail only the case of A_2 with potential $v(q) = \sin^{-2} q$.

3. Root systems

We give here only basic definitions. For more details, see [7, 34, 35].

Let V be a l-dimensional real vector space with a standard scalar product $(\cdot, \cdot)$, $(\alpha, \beta) = \sum \alpha_j\beta_j$, and let s_α be a reflection in the hyperplane through the origin orthogonal to the vector α,

$$s_\alpha q = q - (q, \alpha^\vee)\alpha \qquad \alpha^\vee = \frac{2}{(\alpha, \alpha)}\alpha. \tag{3.1}$$

Consider a finite set of non-zero vectors $R = \{\alpha\}$ generating V and satisfying the following conditions:

(1) For any $\alpha \in R$, the reflection s_α conserves $R : s_\alpha R = R$.
(2) For all $\alpha, \beta \in R$, we have $(\alpha^\vee, \beta) \in \mathbb{Z}$.

The set $\{s_\alpha\}$ generates the finite group $W(R)$ which is called the Weyl group of R. The root system R is called a *reduced* one only if $\pm\alpha$ are vectors collinear to α in R.

Let us choose the hyperplane which does not contain any root. This hyperplane divides the root system into two subsets, $R = R^+ \bigcup R^-$, where R^+ is called the set of positive roots.

† In some papers, the name Jack polynomials [4] is used. However, Jack polynomials are a very special case of the polynomials under consideration. Therefore, we prefer to use the name generalized Gegenbauer polynomials for the general case and Jack polynomials for the special case.

In R^+, there is the basis $\{\alpha_1, \ldots, \alpha_l\}$ such that $\alpha = \sum_j n_j\alpha_j, n_j \geqslant 0$, for any $\alpha \in R^+$. This is called the set of simple roots. The root system R is called *irreducible* if it cannot decompose into two non-empty subsets R_1 and R_2 which are orthogonal to each other.

Let $\{\alpha_1, \ldots, \alpha_l\}$ be the set of simple roots, let R^+ be the set of positive roots, and $\{\lambda_j\}$ be a dual basis or the basis of fundamental weights: $(\lambda_j, \alpha_k) = \delta_{jk}$.

Let Q be the root lattice, and Q^+ be the cone of positive roots:

$$Q = \left\{\beta : \beta = \sum_{j=1}^{l} m_j\alpha_j,\ m_j \in \mathbb{Z}\right\} \qquad Q^+ = \left\{\gamma : \gamma = \sum_{j=1}^{l} n_j\alpha_j,\ n_j \in \mathbb{N}\right\}. \tag{3.2}$$

Let P be a weight lattice, and P^+ be a cone of dominant weights:

$$P = \left\{\lambda : \lambda = \sum_{j=1}^{l} m_j\lambda_j,\ m_j \in \mathbb{Z}\right\} \qquad P^+ = \left\{\mu : \mu = \sum_{j=1}^{l} n_j\lambda_j,\ n_j \in \mathbb{N}\right\}. \tag{3.3}$$

Following [7, 18], we define a partial order on P as follows: $\lambda \geqslant \mu$ if and only if $\lambda - \mu \in Q^+$, or $(\lambda, \lambda_j) \geqslant (\mu, \lambda_j)$ for all $j = 1, \ldots, l$. The set of linear combinations over $\mathbb{R}$ of functions $f_\lambda(q) = \exp\{2\mathrm{i}(\lambda, q)\}$, $\lambda \in P, q \in V$, may be considered as the group algebra A over $\mathbb{R}$ of the free Abelian group P. For any $\lambda \in P$, we denote the corresponding element of A as $\mathrm{e}^\lambda \sim f_\lambda(q)$. So, $\mathrm{e}^\lambda\mathrm{e}^\mu = \mathrm{e}^{\lambda+\mu}$, $(\mathrm{e}^\lambda)^{-1} = \mathrm{e}^{-\lambda}$, and the identity element of A as $\mathrm{e}^0 = 1$. Then e^λ, $\lambda \in P$, form an $\mathbb{R}$-basis of A.

The Weyl group $W(R)$ acts on P and hence also on A: $s(\mathrm{e}^\lambda) = \mathrm{e}^{s\lambda}$ for $s \in W$ and $\lambda \in P$. Let A^W denotes the subalgebra of W-invariant elements of A. It is evident that each W-orbit in P contains only one point in P^+. Consequently, the monomial symmetric functions

$$m_\lambda = \sum_{\mu \in W\cdot\lambda} \mathrm{e}^\mu \qquad \lambda \in P^+ \tag{3.4}$$

form the $\mathbb{R}$-basis of A^W.

4. The Clebsch–Gordan series

Let us recall the main results of [5] and specialize them to the A_2 case with potential $v(q) = \sin^{-2} q$.

The Schrödinger equation for this quantum system has the form

$$H\Psi^\kappa = E(\kappa)\Psi^\kappa \qquad H = -\Delta_2 + U(q_1, q_2, q_3) \qquad \Delta_2 = \sum_{j=1}^{3} \frac{\partial^2}{\partial q_j^2} \tag{4.1}$$

with potential

$$U(q_1, q_2, q_3) = \kappa(\kappa - 1)(\sin^{-2}(q_1 - q_2) + \sin^{-2}(q_2 - q_3) + \sin^{-2}(q_3 - q_1)). \tag{4.2}$$

The ground state wavefunction and its energy are

$$\Psi_0^\kappa(q) = \left(\prod_{j<k}^{3} \sin(q_j - q_k)\right)^\kappa \qquad E_0(\kappa) = 8\kappa^2. \tag{4.3}$$

Substituting $\Psi_\lambda^\kappa = \Phi_\lambda^\kappa\Psi_0^\kappa$ in (4.1), we obtain

$$-\Delta^\kappa\Phi_\lambda^\kappa = \varepsilon_\lambda(\kappa)\Phi_\lambda^\kappa \qquad \Delta^\kappa = \Delta_2 + \Delta_1^\kappa \qquad \varepsilon_\lambda(\kappa) = E_\lambda(\kappa) - E_0(\kappa). \tag{4.4}$$

Here the operator Δ_1^κ takes the form

$$\Delta_1^\kappa = \kappa \sum_{j<k}^{3} \cot(q_j - q_k)\left(\frac{\partial}{\partial q_j} - \frac{\partial}{\partial q_k}\right). \tag{4.5}$$

It is easy to see that the set of symmetric polynomials in variables $\exp(2iq_j)$ is invariant under the action of Δ^κ. Such polynomial m_λ is labelled by the $SU(3)$ highest weight $\lambda = m\lambda_1 + n\lambda_2$, with m, n being non-negative integers, and $\lambda_{1,2}$ being two fundamental weights. In general,

$$\Phi_\lambda^\kappa = \sum_{\mu \leqslant \lambda} C_\lambda^\mu(\kappa) m_\mu \qquad \mu, \lambda \in P^+ \qquad m_\mu = \sum_{\nu \in W\cdot\mu} e^{2i(q,\nu)} \tag{4.6}$$

where P^+ denotes the cone of dominant weights, W is the Weyl group, and $C_\lambda^\mu(\kappa)$ are some constants.

As was shown in [5], the product of two wavefunctions is a finite sum of wavefunctions (a sort of κ-deformed Clebsch–Gordan series):

$$\Phi_\mu^\kappa \, \Phi_\lambda^\kappa = \sum_{\nu \in D_\mu(\lambda)} C_{\mu\lambda}^\nu(\kappa) \Phi_\nu^\kappa. \tag{4.7}$$

In this equation, $D_\mu(\lambda) = (D_\mu + \lambda) \cap P^+$, where D_μ is a weight diagram of the representation with the highest weight μ.

Since Φ_μ^κ are symmetric functions of $\exp(2iq_j)$, it is convenient to use a new set of variables:

$$\begin{aligned} z_1 &= e^{2iq_1} + e^{2iq_2} + e^{2iq_3} \\ z_2 &= e^{2i(q_1+q_2)} + e^{2i(q_2+q_3)} + e^{2i(q_3+q_1)} \\ z_3 &= e^{2i(q_1+q_2+q_3)}. \end{aligned} \tag{4.8}$$

In the centre-of-mass frame ($\sum_i q_i = 0$), the wavefunctions depend only on two variables chosen as z_1 and z_2 (in this case, $z_3 = 1$). In these variables, up to a normalization factor, we have

$$\Delta^\kappa = (z_1^2 - 3z_2)\partial_1^2 + (z_2^2 - 3z_1)\partial_2^2 + (z_1 z_2 - 9)\partial_1\partial_2 + (3\kappa + 1)(z_1\partial_1 + z_2\partial_2) \tag{4.9}$$

where $\partial_i = \partial/\partial z_i$. Corresponding eigenvalues are

$$\varepsilon_{m,n}(\kappa) = m^2 + n^2 + mn + 3\kappa(m+n). \tag{4.10}$$

We shall use the normalization for polynomials Φ_λ^κ such that the coefficient at the highest monomial is equal to one. Denoting them by $P_{m,n}^\kappa$, we have

$$P_{m,n}^\kappa(z_1, z_2) = \sum_{p,q} C_{m,n}^{p,q}(\kappa) z_1^p z_2^q = z_1^m z_2^n + \text{lower terms} \tag{4.11}$$

with $p + q \geqslant m + n$ and $p - q \equiv m - n \pmod 3$. As it is easy to see, the first polynomials are

$$P_{0,0}^\kappa = 1 \qquad P_{1,0}^\kappa = z_1 \qquad P_{0,1}^\kappa = z_2. \tag{4.12}$$

Simple consequences of (4.7) for $P_\lambda^\kappa = P_{1,0}^\kappa$ or $P_{0,1}^\kappa$ are [5]

$$\begin{aligned} z_1 \, P_{m,n}^\kappa &= P_{m+1,n}^\kappa + a_{m,n}(\kappa) P_{m,n-1}^\kappa + c_m(\kappa) P_{m-1,n+1}^\kappa \\ z_2 \, P_{m,n}^\kappa &= P_{m,n+1}^\kappa + \tilde{a}_{m,n}(\kappa) P_{m-1,n}^\kappa + c_n(\kappa) P_{m+1,n-1}^\kappa \end{aligned} \tag{4.13}$$

where

$$\begin{aligned} a_{m,n}(\kappa) &= \tilde{a}_{n,m}(\kappa) = c_n(\kappa) c_{m+n+\kappa}(\kappa) \\ c_m(\kappa) &= \frac{e(m)}{e(\kappa+m)} \qquad e(m) = \frac{m}{m-1+\kappa}. \end{aligned} \tag{4.14}$$

Below we shall construct such polynomials using these recurrence relations.

5. A_2 case

Now we proceed to the case of $A_2 \sim su(3)$. In this case, the representation d of A_2 is characterized by two non-negative numbers $d = d_{mn}$.

We start with the Clebsch–Gordan series

$$\begin{aligned} d_{10} \otimes d_{n+1,0} &= d_{n+2,0} \oplus d_{n,1} \\ d_{01} \otimes d_{n0} &= d_{n,1} \oplus d_{n-1,0}. \end{aligned} \tag{5.1}$$

Excluding $d_{n,1}$, we obtain

$$d_{n-1,0} \ominus (d_{01} \otimes d_{n0}) \oplus (d_{10} \otimes d_{n+1,0}) \ominus d_{n+2,0} = 0$$

or

$$\chi_{n-1,0} - z_2\chi_{n,0} + z_1\chi_{n+1,0} - \chi_{n+2,0} = 0 \tag{5.2}$$

where the following notations are introduced:

$$\begin{aligned} z_1 &= \chi_{10} = \mathrm{e}^{\mathrm{i}\theta_1} + \mathrm{e}^{\mathrm{i}\theta_2} + \mathrm{e}^{\mathrm{i}\theta_3}, \\ z_2 &= \chi_{01} = \mathrm{e}^{-\mathrm{i}\theta_1} + \mathrm{e}^{-\mathrm{i}\theta_2} + \mathrm{e}^{-\mathrm{i}\theta_3}. \end{aligned} \tag{5.3}$$

From this, we obtain the expression for the generating function

$$\begin{aligned} F_0^1(z_1, z_2; u) &= \sum_{n=0}^{\infty} \chi_{n0}(z_1, z_2)u^n \\ F_0^1(z_1, z_2; u) &= (1 - z_1u + z_2u^2 - u^3)^{-1}. \end{aligned} \tag{5.4}$$

Let us define now the κ-deformed functions $\tilde{P}^{\kappa}_{n,0}(z_1, z_2)$ by the formula

$$F^{\kappa}(z_1, z_2; u) = (1 - z_1u + z_2u^2 - u^3)^{-\kappa} = \sum_{n=0}^{\infty} \tilde{P}^{\kappa}_{n,0}(z_1, z_2)u^n. \tag{5.5}$$

Analogously to A_1 case, we may obtain all other formulae from this one.

Differentiating F^{κ} on u, z_1 and z_2, we get

$$\begin{aligned} &F^{\kappa}_u = \kappa\left(z_1 - 2z_2\,u + 3u^2\right) F^{\kappa+1} \\ &F^{\kappa}_{z_1,z_1} = \kappa(\kappa+1)u^2\, F^{\kappa+2} \qquad F^{\kappa}_{z_2,z_2} = \kappa(\kappa+1)u^4\, F^{\kappa+2} \\ &F^{\kappa}_{z_1,z_2} = -\kappa(\kappa+1)u^3 F^{\kappa+2} \qquad uF^{\kappa}_u = \kappa(z_1u - 2z_2u^2 + 3u^3)F^{\kappa+1} \end{aligned}$$

and

$$\begin{aligned} D_u^2 F^{\kappa} = \{&\kappa(z_1u - 4\,z_2u^2 + 9u^3)(1 - z_1u + z_2u^2 - u^3) \\ &+\kappa(\kappa+1)(z_1u - 2z_2u^2 + 3u^3)^2\}F^{\kappa+2} \end{aligned}$$

$$D_u = u\partial_u$$

or

$$\begin{aligned} &(1 - z_1u + z_2u^2 - u^3)F^{\kappa}_u = \kappa(z_1 - 2z_2u + 3u^2)F^{\kappa} \\ &(n+3)\tilde{P}^{\kappa}_{n+3} - z_1(n+2)\tilde{P}^{\kappa}_{n+2} + z_2(n+1)\tilde{P}^{\kappa}_{n+1} - n\tilde{P}^{\kappa}_n = \kappa z_1\tilde{P}^{\kappa}_{n+2} - 2\kappa z_2\tilde{P}^{\kappa}_{n+1} + 3\kappa\tilde{P}^{\kappa}_n. \end{aligned} \tag{5.6}$$

So, we obtain the important recurrence formula

$$(n+3)\tilde{P}^{\kappa}_{n+3,0} = (n+2+\kappa)z_1\tilde{P}^{\kappa}_{n+2,0} - (n+1+2\kappa)z_2\tilde{P}^{\kappa}_{n+1,0} + (n+3\kappa)\tilde{P}_{n,0}. \tag{5.7}$$

Now let us differentiate F^{κ} on z_1. We have

$$F^{\kappa}_{z_1} = \kappa u F^{\kappa+1}. \tag{5.8}$$

Hence

$$\partial_{z_1} P^{\kappa}_{n,0} = n P^{\kappa+1}_{n-1,0}. \tag{5.9}$$

Analogously,

$$F^{\kappa}_{z_2} = -\kappa u^2 F^{\kappa+1}. \tag{5.10}$$

Therefore,

$$\partial_{z_2} P^{\kappa}_{n,0} = -\frac{n(n-1)}{\kappa+n-1} P^{\kappa+1}_{n-2,0}.$$

Finally, we have the basic differential equation for $F^{\kappa}(z_1, z_2;\ u)$:

$$\begin{aligned}((D^2_{z_1} + D^2_{z_2} + D_{z_1} D_{z_2}) - 3z_2\partial^2_{z_1} - 3z_1\partial^2_{z_2} - 9\partial_{z_1}\partial_{z_2} + 3\kappa(D_{z_1} + D_{z_2}))F^{\kappa} \\ = (D^2_u + 3\kappa D_u)F^{\kappa}(z_1, z_2; u) \qquad D_{z_1} = z_1\partial_{z_1} \quad D_{z_2} = z_2\partial z_2 \quad D_u = u\partial_u.\end{aligned}$$

Let us note that the normalization of polynomials $\tilde{P}^{\kappa}_{n,0}(z_1, z_2)$ follows from the expression (5.5) for the generating function. Namely,

$$\tilde{P}^{\kappa}_{n0}(z_1, z_2) = \frac{(\kappa)_n}{n!} z_1^n + \cdots = \frac{(\kappa)_n}{n!} P^{\kappa}_{n,0}(z_1, z_2) \tag{5.11}$$

where

$$(\kappa)_n = (\kappa)(\kappa+1)\ldots(\kappa+n-1).$$

The main property of this normalization is that $\tilde{P}^{\kappa}_{n,0}(z_1, z_2)$ has a polynomial dependence on the parameter κ.

Now we shall consider other Clebsch–Gordan series for $\kappa = 1$:

$$d_{1,0} \otimes d_{n+1,0} = d_{n+2,0} \oplus d_{n,1}.$$

According to [5], the analogous formula is valid for an arbitrary value of κ, i.e.,

$$a_n z_1 \tilde{P}^{\kappa}_{n+1,0} = b_n \tilde{P}^{\kappa}_{n+2,0} + c_n \tilde{P}^{\kappa}_{n,1} \tag{5.12}$$

where coefficients a_n, b_n and c_n do not depend on z_1 and z_2 but may depend on κ. Comparing coefficients at z_1^{n+2} yields

$$a_n = \kappa + n + 1 \qquad b_n = n + 2.$$

From this relation, we may determine the function $\tilde{P}_{n,1}$ up to the normalizing constant c_n:

$$c_n(\kappa)\tilde{P}^{\kappa}_{n,1} = (\kappa+n+1)z_1\tilde{P}^{\kappa}_{n+1,0} - (n+2)\tilde{P}^{\kappa}_{n+2,0}. \tag{5.13}$$

Let us now calculate the generating function for both left- and right-hand sides of this equation,

$$\begin{aligned}G^{\kappa} &= \sum_{n=0}^{\infty} c_n(\kappa)\, \tilde{P}^{\kappa}_{n,1}(z_1, z_2) \\ G^{\kappa} &= \frac{\kappa z_1}{u}(F^{\kappa}_0 - 1) + \frac{z_1}{u} F^{\kappa}_1 - \frac{1}{u^2}(F^{\kappa}_1 - \kappa z_1 u)\end{aligned} \tag{5.14}$$

where

$$F^{\kappa}_0 = \sum_{n=0}^{\infty} \tilde{P}^{\kappa}_{n,0} u^n \qquad F^{\kappa}_1 = \sum_{n=0}^{\infty} n\tilde{P}^{\kappa}_{n,0} u^n = D_u F^{\kappa}_0$$

and

$$G^{\kappa} = \frac{1}{u^2}(\kappa z_1 u - (1 - z_1 u)D_u)F^{\kappa}_0.$$

Finally,

$$G^\kappa = \kappa(2z_2 - (z_1z_2+3)u + 2z_1u^2)F_0^{\kappa+1}. \tag{5.15}$$

From this, follows the three-term recurrence relation

$$\tilde{P}^\kappa_{n,1} = \kappa(2z_2\tilde{P}^{\kappa+1}_{n,0} - (z_1z_2+3)\tilde{P}^{\kappa+1}_{n-1,0} + 2z_1\tilde{P}^{\kappa+1}_{n-2,0}).$$

Let us give also the explicit expression for $\tilde{P}^\kappa_{n,0}(z_1,z_2)$ and $\tilde{P}^\kappa_{n,1}(z_1,z_2)$ in other variables

$$z_1 = \mathrm{e}^{\mathrm{i}\theta_1} + \mathrm{e}^{\mathrm{i}\theta_2} + \mathrm{e}^{\mathrm{i}\theta_3} = x_1 + x_2 + x_3$$
$$z_2 = \mathrm{e}^{-\mathrm{i}\theta_1} + \mathrm{e}^{-\mathrm{i}\theta_2} + \mathrm{e}^{-\mathrm{i}\theta_3} = x_1x_2 + x_2x_3 + x_3x_1.$$

As was shown in [22],

$$\tilde{P}^\kappa_{n,0} = \sum_{m_1,m_2,m_3} C^{n,0}_{m_1,m_2,m_3}(\kappa)x_1^{m_1}x_2^{m_2}x_3^{m_3} \qquad m_1+m_2+m_3 = n \qquad x_1x_2x_3 = 1 \tag{5.16}$$

where quantities $C^{n,0}_{m_1,m_2,m_3}(\kappa)$ are κ-deformed three-nomial coefficients

$$C^{n,0}_{m_1,m_2,m_3}(\kappa) = \frac{n!}{m_1!m_2!m_3!}\frac{(\kappa)_{m_1}(\kappa)_{m_2}(\kappa)_{m_3}}{(\kappa)_n}. \tag{5.17}$$

The function $\tilde{P}^\kappa_{n,1}$ may be defined by the formula

$$\begin{aligned}\tilde{P}^\kappa_{n-2,1}(z_1,z_2) &= z_1\tilde{P}^\kappa_{n-1,0} - \tilde{P}_{n,0}\\ \tilde{P}^\kappa_{n-2,1}(z_1,z_2) &= \sum_{m_1,m_2,m_3} C^{n,1}_{m_1,m_2,m_3}(\kappa)x_1^{m_1}x_2^{m_2}x_3^{m_3}.\end{aligned} \tag{5.18}$$

We obtain

$$C^{n,1}_{m_1,m_2,m_3}(\kappa) = C^{n-1,0}_{m_1-1,m_2,m_3}(\kappa) + C^{n-1,0}_{m_1,m_2-1,m_3}(\kappa) + C^{n-1,0}_{m_1,m_2,m_3-1} - C^{n,0}_{m_1,m_2,m_3}(\kappa). \tag{5.19}$$

Substituting the explicit expression for $C^{n,0}_{m_1,m_2,m_3}(\kappa)$, we get

$$C^{n,1}_{m_1,m_2,m_3}(\kappa) = C^{n,0}_{m_1,m_2,m_3}(\kappa)S^{n,1}_{m_1,m_2,m_3} \tag{5.20}$$

where

$$S^{n,1}_{m_1,m_2,m_3} = \frac{\kappa-1+n}{n}\left(\frac{m_1}{\kappa-1+m_1} + \frac{m_2}{\kappa-1+m_2} + \frac{m_3}{\kappa-1+m_3} - \frac{n}{\kappa-1+n}\right). \tag{5.21}$$

At $\kappa = 1$, the quantity $S^{n,1}_{m_1,m_2,m_3}(\kappa)$ is equal to two. As $\kappa \to \infty$, it has an order κ^{-1} and is a symmetric function of m_1, m_2, m_3. So, it should have a form

$$S^{n,1}_{m_1,m_2,m_3} = \frac{\alpha(\kappa-1)^2 + \beta(\kappa-1) + \gamma}{(\kappa-1+m_1)(\kappa-1+m_2)(\kappa-1+m_3)}. \tag{5.22}$$

Now let us follow [6] and construct the general polynomials in terms of the simplest polynomials (Jack polynomials) $P^\kappa_{m,0}$ and $P^\kappa_{0,n}$. We get

$$P^\kappa_{m,0}P^\kappa_{0,n} = \sum_{i=0}^{\min(m,n)} \gamma^i_{m,n}P^\kappa_{m-i,n-i} \tag{5.23}$$

where $\gamma^i_{m,n}$ are given by the explicit expression†

$$\gamma^i_{m,n} = \frac{(\kappa)^i(m)_i(n)_i(3\kappa+m+n-1-i)_i}{i!(\kappa+m-1)_i(\kappa+n-1)_i(2\kappa+m+n-i)_i} \tag{5.24}$$

where

$$\begin{aligned}(x)^i &= x(x+1)\ldots(x+i-1)\\ (x)_i &= x(x-1)\ldots(x-i+1).\end{aligned} \tag{5.25}$$

The constructive aspect of this formula is in its inverted form.

† Note that this expression for $\gamma^i_{m,n}$ may be obtained from the general Macdonald formula [36]; however, the method of proof given in [6] is more convenient here.

Theorem 1 ([6]). *The generalized Gegenbauer polynomials $P^{\kappa}_{m,n}$ of type A_2 are given by the formula*

$$P^{\kappa}_{m,n} = \sum_{i=0}^{\min(m,n)} \beta^{i}_{m,n} P^{\kappa}_{m-i,0} P^{\kappa}_{0,n-i} \tag{5.26}$$

where the constants $\beta^{i}_{m,n}$ are

$$\beta^{i}_{m,n} = \frac{(-1)^i}{i!} \frac{3\kappa+m+n-2i}{3\kappa+m+n-i} \frac{(m)_i (n)_i (\kappa)_i (3\kappa+m+n-1)_i}{(\kappa+m-1)_i (\kappa+n-1)_i (2\kappa+m+n-1)_i}. \tag{5.27}$$

Note that $\beta^{i}_{m,n}$ are obtained by use of the relation

$$\beta^{i}_{m,n} = -\sum_{j=0}^{i-1} \beta^{j}_{m,n} \gamma^{i-j}_{m-j,n-j}. \tag{5.28}$$

From this theorem, we see that the construction of a general polynomial $P^{\kappa}_{m,n}$ is similar to the construction of $SU(3)$ representations from tensor products of two fundamental representations.

Likewise, we can consider other types of decompositions, such as

$$P^{\kappa}_{m,0} P^{\kappa}_{n,0} = \sum_{i=0}^{\min(m,n)} \tilde{\gamma}^{i}_{m,n} P^{\kappa}_{m+n-2i,i}. \tag{5.29}$$

The proof is analogous to (5.23) (see the footnote on page 7). The coefficients $\tilde{\gamma}^{i}_{m,n}$ are given by the formula

$$\tilde{\gamma}^{i}_{m,n} = \frac{(\kappa)^i \, (m)_i \, (n)_i (2\kappa+m+n-1-i)_i}{i!(\kappa+m-1)_i (\kappa+n-1)_i (\kappa+m+n-i)_i}. \tag{5.30}$$

Theorem 2 ([6]). *There is another formula for polynomials $P^{\kappa}_{m,n}$ at $m \geqslant n$:*

$$\tilde{\gamma}^{n}_{m+n,n} P^{\kappa}_{m,n} = \sum_{i=0}^{n} \tilde{\beta}^{i}_{m,n} P^{\kappa}_{m+n+i,0} P^{\kappa}_{n-i,0} \tag{5.31}$$

where

$$\tilde{\beta}^{i}_{m,n} = \frac{(-1)^i}{i!} (\kappa)_i \frac{(m+2i)}{m} \frac{(\kappa+m+n)^i}{(m+n+1)^i} \frac{(m)^i}{(\kappa+m+1)^i} \frac{(n)_i}{(\kappa+n-1)_i}. \tag{5.32}$$

This theorem follows directly from equation (5.29). The coefficients $\tilde{\beta}^{i}_{m,n}$ are found by use of the relation

$$\tilde{\beta}^{i}_{m,n} = -(\tilde{\gamma}^{n-i}_{m+n+i,n-i})^{-1} \sum_{j=0}^{i-1} \tilde{\beta}^{j}_{m,n} \tilde{\gamma}^{n-i}_{m+n+j,n-j}. \tag{5.33}$$

As a by-product, let us specialize equation (5.26) to the case $\kappa = 1$, where $P^{\kappa}_{m,n}$ are nothing but the $SU(3)$ characters. We get

$$P^{1}_{m,n} = P^{1}_{m,0} P^{1}_{0,n} - P^{1}_{m-1,0} P^{1}_{0,n-1}. \tag{5.34}$$

From this, we easily deduce the generating function for $SU(3)$ characters (see, for example, [37])

$$G^1(u,v) = \sum_{m,n=0}^{\infty} u^m v^n P^{1}_{m,n} = \frac{1-uv}{(1-z_1 u + z_2 u^2 - u^3)(1 - z_2 v + z_1 v^2 - v^3)}. \tag{5.35}$$

6. The integral representation for the case of $N = 3$

The integral representation for the case of $N = 2$ coincides with the integral representation for the Gegenbauer polynomials and is well known (see, for example, [14]).

For the special values of κ ($\kappa = \frac{1}{2}, 1, 2, 4$), wavefunctions are related to zonal spherical functions. The integral representation of these functions was obtained by Harish-Chandra by integrating on the some Lie group K [12, 13].

Following [32] let us now consider the A_2 case ($N = 3$) for $\kappa = \frac{1}{2}$, $K = SO(3, \mathbb{R})$. The element $k \in K = SO(3, \mathbb{R})$ may be represented by three unit vectors orthogonal to each other

$$n, l, m \qquad n^2 = l^2 = m^2 = 1 \qquad (n, l) = (l, m) = (m, n) = 0.$$

Here the integral representation for zonal spherical polynomials has the form

$$\Phi_{pq}(x) = \int_K [\Xi_1(x_j; n)]^p [\Xi_2(x_j; n, l)]^q \, \mathrm{d}\mu(n, l) \qquad x_j = \mathrm{e}^{\mathrm{i}q_j} \tag{6.1}$$

where

$$\Xi_1(x_j; n) = n_1^2 x_1 + n_2^2 x_2 + n_3^2 x_3 \qquad \Xi_2(x_j; n, l) = \sum_{j<k} (n_j l_k - n_k l_j)^2 x_j x_k$$

and the integration is taken on the orthogonal group $K = SO(3, \mathbb{R})$, which is equivalent to the space of two unit orthogonal vectors n and l.

Noting that $m_k = \epsilon_{kij} n_i l_j$, we also have $x_1 x_2 = x_3^{-1}, \ldots$. Hence Φ_{pq} is given by (6.1) where

$$\Xi_2(x_j; n, l) = m_1^2 x_1^{-1} + m_2 x_2^{-1} + m_3 x_3^{-1}. \tag{6.2}$$

For vectors n and m, the standard parametrization through the Euler angles φ, θ and ψ may be used:

$$\begin{aligned} &n = (\cos\varphi \sin\theta, \sin\varphi \sin\theta, \cos\theta) \qquad m = \cos\psi \cdot a + \sin\psi \cdot b \\ &a = (-\sin\varphi, \cos\varphi, 0) \qquad b = (-\cos\varphi\cos\theta, -\sin\varphi\cos\theta, \sin\theta) \end{aligned} \tag{6.3}$$

with $\mathrm{d}\mu(k) = \mathrm{d}\mu(n, m) = A \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi \, \mathrm{d}\psi$. Hence in the preceding expression we have a three-dimensional integral which may be calculated by using of the generating functions.

7. The generating function for the case of $N = 3$

Following [32] let us define the generating function of zonal spherical polynomials for $\kappa = \frac{1}{2}$ by the formula

$$F(x_1, x_2, x_3; t_1, t_2) = \sum \Phi_{l_1, l_2}(x_1, x_2, x_3) t_1^{l_1} t_2^{l_2}. \tag{7.1}$$

Then we obtain the integral representation

$$F(x_1, x_2, x_3; t_1, t_2) = \int \left[\prod_{j=1}^{2} (1 - \Xi_j(x; n, \psi) t_j) \right]^{-1} \mathrm{d}\mu(n) \, \mathrm{d}\mu(\psi). \tag{7.2}$$

Let a and b be two unit orthogonal vectors in two-dimensional plane orthogonal to the vector n. Then on this plane an arbitrary unit vector m has the form $\cos\psi \cdot a + \sin\psi \cdot b$. Integrating first on $\mathrm{d}\mu(\psi)$ we get

$$F(x_1, x_2, x_3; t_1, t_2) = \int B^{-1}(n) C^{-1/2}(n) \, \mathrm{d}\mu(n) \qquad \int \mathrm{d}\mu(n) = 1 \tag{7.3}$$

where

$$B = 1 - (n_1^2 x_1 + n_2^2 x_2 + n_3^2 x_3) t_1 \qquad C = (1 - x_2^{-1} t_2)(1 - x_3^{-1} t_2) n_1^2 + \cdots. \tag{7.4}$$

The crucial step for the further integration is the use of the formula

$$B^{-1}C^{-1/2} = \int_0^1 \mathrm{d}\xi\, [B(1-\xi^2) + C\xi^2]^{-3/2}. \tag{7.5}$$

Using this, we obtain

$$F(x_1, x_2, x_3; t_1, t_2) = \int_0^1 \mathrm{d}\xi \int [E(x_1, x_2, x_3; t_1, t_2, n, \xi)]^{-3/2}\, \mathrm{d}\mu(n) \tag{7.6}$$

where

$$E(x_1, x_2, x_3; t_1, t_2, n, \xi) = \sum_j e_j(x_1, x_2, x_3; t_1, t_2, \xi) n_j^2. \tag{7.7}$$

We can now integrate on $\mathrm{d}\mu(n)$. Finally we obtain the one-dimensional integral representation for the generating function

$$F(x_1, x_2, x_3; t_1, t_2) = \int_0^1 \mathrm{d}\xi\, [H(x_1, x_2, x_3; t_1, t_2, \xi)]^{-1/2} \tag{7.8}$$

where $H = h_1 h_2 h_3$, and functions $h_j(\xi; t_1, t_2)$ are given by formulae

$$h_j(\xi; t_1, t_2) = 1 - d_j(t_1, t_2)(1-\xi^2) \qquad d_j(t_1, t_2) = (x_j t_1 + x_j^{-1} t_2 - t_1 t_2). \tag{7.9}$$

From this it follows that if $z_1 = x_1 + x_2 + x_3$ and $z_2 = x_1x_2 + x_2x_3 + x_3x_1$, then

$$\begin{aligned} H = a_0^3 - a_0^2\, [z_1\tau_1 + z_2\tau_2] + a_0[z_2\tau_1^2 + z_1\tau_2^2 + (z_1z_2 - 3)\tau_1\tau_2] \\ -[\tau_1^3 + \tau_2^3 + \tau_1\tau_2[(z_2^2 - 2z_1)\tau_1 + (z_1^2 - 2z_2)\tau_2]] \end{aligned} \tag{7.10}$$

where $a_0 = 1 + (1-\xi^2)t_1t_2$, $\tau_1 = (1-\xi^2)t_1$, $\tau_2 = (1-\xi^2)t_2$. Note that from (7.10) it follows that integral (7.8) is an elliptic one and may be expressed in terms of standard elliptic integrals.

Expanding $F(x_1, x_2, x_3; t_1, t_2)$ in power series of the variable t_2 we obtain

$$F(x_1, x_2, x_3; t_1, t_2) = \sum_{q=0}^{\infty} F_q(x_1, x_2, x_3; t_1) t_2^q. \tag{7.11}$$

We have

$$F_0(x_1, x_2, x_3; t) = \int_0^1 \mathrm{d}\xi\, [H_0]^{-1/2} \tag{7.12}$$

and

$$F_1 = \tfrac{1}{2} \int_0^1 \mathrm{d}\xi\, H_1 [H_0]^{-3/2} \tag{7.13}$$

where

$$H_0 = 1 - z_1\tau_1 + z_2\,\tau_1^2 - \tau_1^3$$
$$H_1 = (1-\xi^2)z_2 - [3\xi^2 + z_1z_2(1-\xi^2)]\tau_1 + [2z_1\xi^2 + (1-\xi^2)z_2^2]\tau_1^2 - z_2\tau_1^3.$$

From integral representation (7.8), many useful formulae may be obtained. Here we give only one of them, namely, as $z_1, z_2 \to \infty$,

$$\Phi_{m,n}(z_1, z_2) \approx A_{m,n} z_1^p z_2^q \qquad A_{m,n} = \frac{(\frac{1}{2})_m (\frac{1}{2})_n}{(1)_m (1)_n} \frac{(1)_{m+n}}{(\frac{3}{2})_{m+n}}. \tag{7.14}$$

One can also show [38] that for arbitrary κ the analogous integral representation is valid

$$F^\kappa(z_1, z_2; t_1, t_2) = \sum_{m,n=1}^{\infty} A_{m,n}(\kappa) P_{m,n}^\kappa(z_1, z_2) t_1^m t_2^n = \int_0^1 [H]^{-\kappa}\, \mathrm{d}\mu^\kappa(\xi) \tag{7.15}$$

where H is given by (7.10), $P_{m,n}^\kappa(z_1, z_2) \sim z_1^m z_2^n$ as $z_1, z_2 \to \infty$, and

$$\mathrm{d}\mu^\kappa(\xi) = [\xi(1-\xi^2)]^{2\kappa-1} \qquad A_{m,n}(\kappa) = \frac{1}{2} \frac{\Gamma(\kappa)\Gamma(2\kappa)}{\Gamma(3\kappa)} \frac{(\kappa)_m (\kappa)_n}{(1)_m (1)_n} \frac{(2\kappa)_{m+n}}{(3\kappa)_{m+n}}. \tag{7.16}$$

Note that for the case under consideration another integral representation for $N = 3$ was obtained in [31] in terms of ${}_3F_2$ hypergeometric series.

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Appendix. Some formulae for Macdonald polynomials for the A_2 case (by Perelomov, Ragoucy and Zaugg)

Let us give first some necessary for us information. For other results and details, see [36].

The Macdonald polynomials of type A_2 may be defined as polynomial eigenfunctions of the Macdonald difference equation

$$\begin{aligned} &M^1 P_{m,n}^{(q,t)}(x_1, x_2, x_3) = \lambda P_{m,n}^{(q,t)}(x_1, x_2, x_3) \qquad P_{m,n} = z_1^m z_2^n + \text{lower terms} \\ &z_1 = x_1 + x_2 + x_3 \qquad z_2 = x_1x_2 + x_2x_3 + x_3x_1 \qquad z_3 = x_1x_2x_3 \end{aligned} \tag{A.1}$$

where

$$M^1 = \prod_{j\neq k} \frac{(tx_j - x_k)}{(x_j - x_k)} T_j \qquad T_1 f(x_1, x_2, x_3) = f(qx_1, x_2, x_3), \ldots. \tag{A.2}$$

The Macdonald polynomials satisfy the following recurrence relation:

$$z_1 P_{m,n}^{(q,t)}(z_1, z_2, z_3) = P_{m+1,n}^{(q,t)} + a_{m,n}(q, t) P_{m,n-1}^{(q,t)} + b_{m,n}(q, t) P_{m-1,n+1}^{(q,t)} \tag{A.3}$$

where

$$\begin{aligned} &a_{mn} = c_n \tilde{c}_{m+n} \qquad b_{mn} = c_m \\ &c_m(q, t) = \frac{(1 - q^m)(1 - t^2 q^{m-1})}{(1 - tq^m)(1 - tq^{m-1})} \\ &\tilde{c}_{m+n}(q, t) = \left(\frac{1 - tq^{m+n}}{1 - t^2 q^{m+n}}\right)\left(\frac{1 - t^3 q^{m+n-1}}{1 - t^2 q^{m+n-1}}\right). \end{aligned} \tag{A.4}$$

Note that

$$c_m(q, 1) = c_m(q, q) = 1 \qquad \tilde{c}_{m+n}(q, 1) = \tilde{c}_{m+n}(q, q) = 1. \tag{A.5}$$

Now let us consider the case $t = q^k$, k being an integer. In this case, the explicit expression for the generating function of polynomials $P_{n,0}^{(k)} \equiv P_{n,0}^{(q,t)}$ has the form

$$\begin{aligned} &G^{(k)}(u) = \prod_{j=0}^{k-1} F^1(q^j u) \\ &G^{(k)}(u) = \sum C_n^k P_{n,0}^{(k)} u^n \\ &C_n^k = \frac{[k]_n}{[1]_n} \qquad [x]_n = [x][x+1]\ldots[x+n-1] \qquad [x] = \frac{1 - q^x}{1 - x}. \end{aligned} \tag{A.6}$$

From this we get a three-term recurrence relation analogous to (5.7)

$$[n+1]\tilde{P}_{n+1}^k = [k+n] z_1 \tilde{P}_n^k - [2k+n-1] z_2 \tilde{P}_{n-1}^k + [3k+n-2] z_3 \tilde{P}_{n-2}^k \tag{A.7}$$

which follows from the recurrence relation

$$(1 - z_1 u + z_2 u^2 - z_3 u^3) G^{(k)}(u) = (1 - z_1 u q^k + z_2 u^2 q^{2k} - z_3 u^3 q^{3k}) G(qu). \tag{A.8}$$

From here, we obtain

$$\tilde{P}_0^\kappa = 1 \qquad \tilde{P}_1^\kappa = [\kappa] z_1 \qquad \tilde{P}_2^\kappa = \frac{[\kappa+1][\kappa]}{[1][2]} z_1^2 - \frac{[2\kappa]}{[2]} z_2, \ldots \qquad \tilde{P}_{m,0} = \frac{[1]_m}{[\kappa]_m} P_{m,0}.$$

Let us also give the formula for the generating function for the case of arbitrary t:

$$G^{(q,t)}(u) = \prod_{j=0}^{\infty} \frac{(1-q^j tu)}{(1-q^j u)} = \sum_{j=0}^{\infty} c_j(q,t)\, u^j. \tag{A.9}$$

Using the results from the book [36], it is not difficult to get the formulae

$$P_{m,0}^{(q,t)} P_{0,n}^{(q,t)} = \sum_{i=0}^{\min(m,n)} \gamma_{m,n}^i P_{m-i,n-i}^{(q,t)} \tag{A.10}$$

where

$$\gamma_{m,n}^i = \frac{(t;q)_i (q^m;q^{-1})_i (q^n;q^{-1})_i (q^{m+n-1-i}t^3;q^{-1})_i}{(q;q)_i (q^{m-1}t;q^{-1})_i (q^{n-1}t;q^{-1})_i (q^{m+n-i}t^2;q^{-1})_i} \tag{A.11}$$

and

$$P_{m,0}^{(q,t)} P_{n,0}^{(q,t)} = \sum_{i=0}^{\min(m,n)} \tilde{\gamma}_{m,n}^i P_{m+n-2i,i}^{(q,t)} \tag{A.12}$$

where

$$\tilde{\gamma}_{m,n}^i = \frac{(t;q)_i (q^m;q^{-1})_i (q^n;q^{-1})_i (q^{m+n-i-1}t^2;q^{-1})_i}{(q;q)_i (q^{m-1}t;q^{-1})_i (q^{n-1}t;q^{-1})_i (q^{m+n-i}t;q^{-1})_i}. \tag{A.13}$$

Inverting formulae (A.10) and (A.11) as in [6], we arrive at the following theorem.

Theorem 1(a). *The Macdonald polynomials $P_{m,n}^{(q,t)}$ of type A_2 are given by the formula*

$$P_{m,n}^{(q,t)} = \sum_{i=0}^{\min(m,n)} \beta_{m,n}^i P_{m-i,0}^{(q,t)} P_{0,n-i}^{(q,t)} \tag{A.14}$$

where constants $\beta_{m,n}^i$ have the form

$$\beta_{m,n}^i = (-1)^i q^{i(i-1)/2} \frac{(t;q^{-1})_i}{(q;q)_i} \frac{1-q^{m+n-2i}t^3}{1-q^{m+n-i}t^3} \frac{(q^m;q^{-1})_i (q^n;q^{-1})_i (q^{m+n-1}t^3;q^{-1})_i}{(q^{m-1}t;q^{-1})_i (q^{n-1}t;q^{-1})_i (q^{m+n-1}t^2;q^{-1})_i}.$$

In similar way, we get the following theorem.

Theorem 2(a). *The Macdonald polynomials $P_{m,n}^{(q,t)}$ of type A_2 are given by the formula*

$$\tilde{\gamma}_{m+n,n}^n P_{m,n}^{(q,t)} = \sum_{i=0}^{n} \tilde{\beta}_{m,n}^i P_{m+n+i,0}^{(q,t)} P_{n-i,0}^{(q,t)} \qquad m \geqslant n \tag{A.15}$$

where constants $\tilde{\gamma}_{m,n}^i$ are given by formula (A.13), and

$$\tilde{\beta}_{m,n}^i = (-1)^i q^{i(i-1)/2} \frac{(t;q^{-1})_i}{(q;q)_i} \frac{(1-q^{m+2i})}{(1-q^m)} \frac{(q^m;q)_i (q^n;q^{-1})_i (q^{m+n}t;q)_i}{(q^{m+n+1};q)_i (q^{m+1}t;q)_i (q^{n-1}t;q^{-1})_i}. \tag{A.16}$$

From theorems 1(a) and 2(a), many interesting identities may be obtained. Here we give one of them:

$$S_{n,l}^k = \sum_{i=0}^{l} (-1)^i q^{i(i-1)/2} \frac{[3k+n-2i]}{[3k+n-i]} \left(\prod_{j=1}^{i} \frac{[k-j+1]}{[j]} \frac{[3k+n-j]}{[2k+n-j]} \right)$$
$$\times \left(\prod_{j=0}^{l-i-1} \frac{[k+j]}{[j+1]} \frac{[3k+n-l-i-j-1]}{[2k+n-l-i-j]} \right) = 0 \tag{A.17}$$

where $[n] = (1-q^n)/(1-q)$.

The simplest version of this formula is

$$\sum_{i=0}^{l}(-1)^i q^{i(i-1)/2}\frac{[k+l-i-1]!}{[i]![l-i]![k-i]!}=0. \tag{A.18}$$

We give below the *list of polynomials* $P_{m,n}^{(q,t)}$ at $m+n\leqslant 4$:

$$P_{0,0}^{(q,t)}=1 \qquad P_{1,0}^{(q,t)}=z_1 \qquad P_{0,1}^{(q,t)}=z_2$$

$$P_{2,0}^{(q,t)}=z_1^2-\frac{(1-q)(1+t)}{(1-qt)}z_2$$

$$P_{1,1}^{(q,t)}=z_1z_2-\frac{(1-q)(1+t+t^2)}{(1-qt^2)}z_3$$

$$P_{3,0}^{(q,t)}=z_1^3-\frac{(1-q)(2+q+t+2qt)}{1-q^2t}z_1z_2+\frac{(1-q)^2(1+q)(1+t+t^2)}{(1-qt)(1-q^2t)}z_3$$

$$P_{2,1}^{(q,t)}=z_1^2z_2-\frac{(1-q)(1+t)}{1-qt}z_2^2-\frac{(1-q^2)(1-qt^3)}{(1-qt)^2(1+qt)}z_1z_3$$

$$\begin{aligned}P_{4,0}^{(q,t)}=z_1^4&-\frac{(1-q)(3+2q+q^2+t+2qt+3q^2t)}{1-q^3t}z_1^2z_2\\&+\frac{(1-q)^2(1+q+q^2)(1+t)(1+qt)}{(1-q^2t)(1-q^3t)}z_2^2\\&+\frac{(1-q)^2(1+q)(2+q+q^2+t+2qt+q^2t+t^2+qt^2+2q^2t^2)}{(1-q^2t)(1-q^3t)}z_1z_3\end{aligned}$$

$$\begin{aligned}P_{3,1}^{(q,t)}=z_1^3z_2&-\frac{(1-q)(2+q+t+2qt)}{1-q^2t}z_1z_2^2-\frac{(1-q^3)(1-q^2t^3)}{(1-q^2t)(1-q^3t^2)}z_1^2z_3\\&+\frac{(1-q)^2}{(1-q^2t)(1-q^3t^2)}\times(2+2q+q^2+2t+4qt+3q^2t+q^3t+t^2\\&+3qt^2+4q^2t^2+2q^3t^2+qt^3+2q^2t^3+2q^3t^3)z_2z_3\end{aligned}$$

$$\begin{aligned}P_{2,2}^{(q,t)}=z_1^2z_2^2&-\frac{(1-q)(1+t)}{1-qt}(z_2^3+z_1^3z_3)\\&-\frac{(1-q)(3q+q^2-3t+2q^2t+q^3t-t^2-2qt^2+3q^3t^2-qt^3-3q^2t^3)}{(1-qt)(1-q^3t^2)}\\&\times z_1z_2z_3\\&+\frac{(1-q)^2(1+q)(1+t+t^2)(q-t+q^2t-qt^2+q^3t^2-q^2t^3)}{(1-qt)(1-q^2t^2)(1-q^3t^2)}z_3^2.\end{aligned}$$

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